Summer term 2018 University of Bonn
Real and harmonic analysis Dr. Pavel Zorin-Kranich
Exercise sheet 12 Dr. Olli Saari

Submit the solutions in groups of two at the lecture on Tuesday, 2018-07-10

Exercise 1. Let $k > 1$ and suppose ϕ is of class C^k in an interval $[a, b]$. Assume that $|\phi^{(k)}(t)| \geq 1$ for all $t \in [a, b]$. Prove

$$\left| \int_a^b e^{i\lambda\phi(t)}\, dt \right| \leq c\lambda^{-1/k}$$

for all $\lambda > 0$ and c independent of a and b. (Hint: use the first exercise of the exercise sheet 10 and induction.)

Exercise 2. Let T be a Calderón–Zygmund operator.

(a) Prove that there is a constant c such that for any L^2-atom a we have $\|Ta\|_1 \leq c$. (Hint: study separately $\int_{2Q} + \int_{(2Q)^c}$ where Q is the support of the atom a. Use L^2 bound in one of them and properties of the kernel in the other one.)

(b) Prove that if $f \in H^1$, then for any atomic decomposition $f = \sum_i \lambda_i a_i$ it holds

$$Tf = \sum_i \lambda_i T a_i$$

almost everywhere. Conclude that $T : H^1 \to L^1$ is bounded. (Hint: show that $|\{|Tf - \sum_i \lambda_i T a_i| > \delta\}| = 0$ for all $\delta > 0$. Split the sum into two parts.)

(c) Let $f \in L^1 \cap L^\infty$. Prove that

$$\|Tf\|_{BMO} \leq c\|f\|_\infty.$$

(Hint: for any cube Q, decompose $f = f1_{2Q} + f1_{(2Q)^c}$ and study the pieces separately)